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Asymptotic behavior of the non-autonomous 3D Navier–Stokes problem with coercive force ☆

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ABSTRACT

We construct pullback attractors to the weak solutions of the three-dimensional Dirichlet problem for the incompressible Navier–Stokes equations in the case when the external force may become unbounded as time goes to plus or minus infinity.

1. Introduction

An *attractor* of a dynamical system is a certain set to which every orbit eventually becomes close. When an autonomous differential equation (or boundary value problem) generates a dynamical system, the corresponding attractor characterizes the long-time behavior of its solutions [23,4,17,24,40]. The study of attractors to the 2D Navier–Stokes equations goes back to Ladyzhenskaya [29], who was followed by lots of authors [2].

The non-autonomous equations do not automatically produce dynamical systems. Instead, one may define an attractor for a *process* (a two-parameter semigroup) related to the solutions of a non-autonomous equation. There are three adequate approaches to this task. The first one is to extend

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the phase space and to deal with the *skew-product* dynamical system [36,23]. The second one [17] is to introduce a concept of a *uniform attractor* which attracts the trajectories uniformly with respect to the time shifts. It turns out that sufficient conditions for existence of a uniform attractor [17,15] guarantee non-emptiness of the set (which is called the *kernel* of the process) of bounded complete trajectories of the process. The *sections* of the kernel possess [15] attraction properties which resemble the ones of the usual attractor of an autonomous system. However, this attraction is not uniform but *pullback*, i.e. it happens when the actual moment of time is fixed and the initial time goes to minus infinity. The pullback mechanism appeared in much earlier works, e.g. [28] (see discussions in [30,22]), but the concept of a *pullback attractor* was proposed by Schmallfuss, Crauel and Flandoli (see [27,18] and the references therein) in the early 1990s, and was since then successfully applied to many systems, e.g. [7,14,26,30,35,47]. This approach turned out to be relevant in much more general situations than the one of [15].[1] Namely, the pullback attractors characterize the behavior of processes rather at each "finite", "present" moment than as time goes to infinity. Therefore, this notion can be used to investigate the limiting behavior of the processes which do not have bounded complete trajectories. Such a situation can arise, for instance, for equations with *coercive*, i.e. unbounded as time goes to plus or minus infinity, right-hand members. The pullback attractors for the 2D Navier–Stokes system with (possibly) coercive non-autonomous body force were constructed in [11,10].

The attractor theory turned out to be generalizable onto the case of the problems which lack the property of uniqueness of solutions (or where such a property remains an open problem). Obviously, such problems do not generate dynamical systems in a normal manner. One of the main motivations for the progress in this direction was the ambition to study the limiting behavior of the weak solutions to the 3D Navier–Stokes problem. There exist several ways of construction of attractors in this case. The first one, based on the theory of multivalued semigroups, goes back to [3], and was developed in [34]. It was used for the weak solutions of the 3D Navier–Stokes problem when the right-hand side is uniformly bounded in H or under an unproved hypothesis [25]. A related *generalized semiflow* approach was proposed in [5], and adapted to stochastic problems in [33].

An alternative method employs the concept of *trajectory attractor*, i.e. the attractor of the translation semigroup in the space of trajectories [37,16]. The sections of the trajectory attractor coincide with a properly defined *global attractor* [41,17]. A similar procedure can be realized in the non-autonomous case at the presence of bounded complete trajectories, generalizing the notions of the uniform attractor and of the kernel [16,41]. The trajectory attractor technique is applicable to the weak solutions of the 3D Navier–Stokes problem [37,16,41,17]. However, it requires the uniform boundedness of the Steklov average (in time) of the square of the V^*-norm of the body force.

The trajectory attractor theory was amplified in [50], where some technical requirements, e.g. the invariance of the trajectory space, were omitted, which allowed us to study some problems where the classical trajectory attractor procedure was not working [43,45,44].

In [6], the attractors to the 3D Navier–Stokes problem were handled in the framework of non-standard analysis.

The treatments of pullback attractors for the non-autonomous problems without uniqueness are predominantly based on the concept of *multivalued dynamical process* [9,12,13,46,32,8]. A trajectory attractor approach was introduced in [20,21], and, in a different manner, in [48]. The framework of [48] does not admit any unbounded trajectories. The considerations of [20,21] were mainly directed at the analysis of stochastic equations; nevertheless, in [21], the deterministic 3D Navier–Stokes problem with unbounded body force was also studied. However, the coercivity was restricted by a complicated condition assuming some "generalized boundedness" as time goes to minus infinity (cf. [21, p. 375]), and the differentiability of the non-autonomous part of the forcing term in the spatial variable was supposed.

In this work, we adapt the ideas from [50] to the pullback attraction case. We introduce the notions of *minimal pullback trajectory $\mathcal{D}$-attractor* and *minimal pullback $\mathcal{D}$-attractor* (note that the latter is not a "trajectory" one). We find some general criteria for existence of these attractors. Then we investigate the relation between our concept of the minimal pullback $\mathcal{D}$-attractor and the existing one of

[1] But the family of kernel sections, in the framework of [15], coincides with the pullback attractor.

the pullback $\mathcal{D}$-attractor for a dynamical process. Finally, we apply this approach to the construction of pullback attractors to the 3D Navier–Stokes problem. The only assumption on the body force is, roughly speaking, that the growth of its V^*-norm at minus infinity can be at most exponential. The same condition was imposed in [11,10] for the 2D model.

The paper is organized as follows. The next section is a preliminary one (notation, etc.). Section 3 is devoted to the general description of our approach to the pullback attractors for the non-autonomous problems without uniqueness. The main results of the section are collected in Subsection 3.2, and the comparison with the pullback $\mathcal{D}$-attractor for a dynamical process is carried out in Subsection 3.4. In the last section, we construct the minimal pullback trajectory $\mathcal{D}$-attractor and the minimal pullback $\mathcal{D}$-attractor for the weak solutions of the three-dimensional incompressible Navier–Stokes problem.

2. Preliminaries and notation

Let Ω be a bounded domain (i.e. an open set, with any kind of boundary) in $\mathbb{R}^3$.

We shall use the standard notations $L_p(\Omega)$, $W_p^\beta(\Omega)$, $H^\beta(\Omega) = W_2^\beta(\Omega)$, $H_0^\beta(\Omega) = \overset{\circ}{W}{}_2^\beta(\Omega)$ $(\beta > 0)$ for the Lebesgue and Sobolev spaces.

Parentheses denote the following bilinear form:

$$(u, v) = \int_\Omega \big(u(x), v(x)\big)_F \, dx,$$

where F is $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^9$ (the space of 3×3-matrices).

The Euclid norm in $\mathbb{R}^3$ is denoted as $|\cdot|$. The symbol $\|\cdot\|$ will stand for the Euclid norm in $L_2(\Omega)$, $L_2(\Omega)^3$, or $L_2(\Omega)^9$. We shall also use the notation $\|v\|_1 = \|\nabla v\|$, $v \in H^1(\Omega)^3$.

Let $\mathcal{V}$ be the set of smooth, divergence-free, compactly supported in Ω functions with values in $\mathbb{R}^3$. The symbols H, V, V_δ $(\delta > 0)$ denote the closures of $\mathcal{V}$ in $L_2(\Omega)^3$, $H^1(\Omega)^3$, $H^\delta(\Omega)^3$, respectively.

Since Ω is bounded, there exists $\lambda_1 > 0$ so that

$$\lambda_1 \|u\|^2 \leqslant \|u\|_1^2, \quad u \in V. \tag{2.1}$$

Following [40], we identify the space H and its conjugate space H^*. Therefore we have the embedding

$$V_\delta \subset H \equiv H^* \subset V_\delta^*.$$

The value of a functional from V_δ^* on an element from V_δ is denoted by brackets $\langle\cdot,\cdot\rangle$. We consider V to be equipped with the norm $\|\cdot\|_1$ and V^* to be equipped with the corresponding norm of a conjugate space.

The symbols $C(\mathcal{J}; E)$, $C_w(\mathcal{J}; E)$, $L_2(\mathcal{J}; E)$, etc. denote the spaces of continuous, weakly continuous, quadratically integrable, etc. functions on an interval $\mathcal{J} \subset \mathbb{R}$ with values in a Banach space E. We recall that a function $u : \mathcal{J} \to E$ is *weakly continuous* if for any linear continuous functional g on E the function $g(u(\cdot)) : \mathcal{J} \to \mathbb{R}$ is continuous. Let us also recall that a pre-norm in the Fréchet space $C([0, +\infty); E)$ may be defined by the formula

$$\|v\|_{C([0,+\infty);E)} = \sum_{i=1}^{+\infty} 2^{-i} \frac{\|v\|_{C([0,i];E)}}{1 + \|v\|_{C([0,i];E)}}.$$

Finally, let us introduce a very trivial notion, which will be useful to simplify the language.

Definition 2.1. A *brochette* over a set $\mathcal{Y}$ is a family of sets $B_t \subset \mathcal{Y}$ depending on a scalar parameter $t \in \mathbb{R}$.

Definition 2.2. For two brochettes B and B^* over $\mathcal{Y}$, we define the intersection $B \cap B^*$ as the family of $(B \cap B^*)_t = B_t \cap B_t^*$, $t \in \mathbb{R}$. We say that B is contained in B^* and write $B \subset B^*$ provided $B_t \subset B_t^*$ for all $t \in \mathbb{R}$.

Remark 2.3. To put it differently, a brochette is a (possibly empty-set-valued) multimap $B : \mathbb{R} \multimap \mathcal{Y}$. Note that the non-empty-set-valued brochettes are sometimes called *non-autonomous sets* [1].

3. Pullback trajectory and global attractors

3.1. Basic definitions

Let E and E_0 be Banach spaces, $E \subset E_0$. Consider an abstract non-autonomous differential equation[2]

$$u'(t) = A\big(t, u(t)\big),$$
$$u : \mathbb{R} \to E, \qquad A : D(A) \to R(A), \quad D(A) = \mathbb{R} \times E_A,\ E_A \subset E. \tag{3.2}$$

We study the limiting behavior of the solutions to (3.2) which continuously depend on time in the topology of E_0.

We denote $\mathcal{T} = C([0, +\infty); E_0) \cap L_{\infty,\mathrm{loc}}(0, +\infty; E)$. Hereafter it is supposed that the space E is reflexive. Then, by a well-known Lions–Magenes lemma, see e.g. [50, Lemma 2.2.6],

$$\mathcal{T} \subset C_w\big([0, +\infty); E\big).$$

Hence, the values of functions from $\mathcal{T}$ belong to E at every time.

We shall use the translation (shift) operators $T(h)$,

$$T(h)(u)(t) = u(t + h),$$

where $h \geqslant 0$ for $u \in \mathcal{T}$, and $h \in \mathbb{R}$ for $u \in C(\mathbb{R}; E_0) \cup L_{\infty,\mathrm{loc}}(\mathbb{R}; E)$.

For every $\tau \in \mathbb{R}$, let us consider some set

$$\mathcal{H}_\tau^+ \subset \mathcal{T}$$

of solutions (strong, weak, etc.) to the shifted equation

$$u'(t) = A\big(t + \tau, u(t)\big), \tag{3.3}$$

on the positive time axis. The sets $\mathcal{H}_\tau^+$ are called *trajectory spaces* and their elements are called *trajectories*. Note that $\mathcal{H}^+$ is a brochette over $\mathcal{T}$ (the *trajectory brochette*).

Remark 3.1. An appropriate trajectory brochette $\mathcal{H}^+$ must be sufficiently "wide" in order to describe well the dynamics of (3.2). Typically, it should be such that for every $a \in E$ and $\tau \in \mathbb{R}$ there exists (but is not necessarily unique) a trajectory $u \in \mathcal{H}_\tau^+$ satisfying the initial condition $u(0) = a$ (cf. [50, Remark 4.2.2] for the autonomous case).

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[2] The symbol "=" may be understood in any appropriate sense (e.g. in the sense of some topological space containing both E and $R(A)$). The derivative "′" may also be considered in any generalized sense. The nonlinear operator A is arbitrary (it may even be multivalued, but in this case the symbol "=" must be replaced by "⊂").

Remark 3.2. As usual in the theory of trajectory attractors, the precise form of Eq. (3.2) is not significant (cf. [17,50]). It merely matters to have a brochette $\mathcal{H}^+$, and everything depends on its properties only. Generally speaking, the nature of $\mathcal{H}^+$ may be different from the one described above.

Now, fix a class $\mathcal{D}$ of brochettes $D = \{D_t \neq \emptyset,\ t \in \mathbb{R}\}$ over E. For each $D \in \mathcal{D}$, let us construct a brochette $\mathcal{H}(D)$ according to the formula

$$\mathcal{H}_t(D) = \left\{u \in \mathcal{H}_t^+ \colon u(0) \in D_t\right\}. \tag{3.4}$$

Definition 3.3. A brochette P over the set $\mathcal{T}$ is called *pullback $\mathcal{D}$-attracting* (for $\mathcal{H}^+$) if for all brochettes $D \in \mathcal{D}$ and $t \in \mathbb{R}$ one has

$$\sup_{u \in \mathcal{H}_\tau(D)} \inf_{v \in P_t} \left\| T(t-\tau)u - v \right\|_{C([0,+\infty);E_0)} \xrightarrow[\tau \to -\infty]{} 0.$$

Remark 3.4. This definition implies that, given a pullback $\mathcal{D}$-attracting brochette P, all the sets P_t are non-empty.

Definition 3.5. A brochette P over $\mathcal{T}$ is called *pullback $\mathcal{D}$-absorbing* (for $\mathcal{H}^+$) if for all $D \in \mathcal{D}$ and $t \in \mathbb{R}$ there is $\tau_0 = \tau_0(D,t) \leqslant t$ such that for all $\tau \leqslant \tau_0$ one has

$$T(t-\tau)\mathcal{H}_\tau(D) \subset P_t,$$

and the function $\tau_0(D,\cdot) \colon \mathbb{R} \to \mathbb{R}$ is non-decreasing for each fixed D.

It is easy to see that any absorbing brochette is an attracting one.

Definition 3.6. A brochette P over $\mathcal{T}$ is called *relatively $\mathcal{T}$-compact* if

(i) P_t is relatively compact in $C([0,+\infty); E_0)$ for every $t \in \mathbb{R}$;
(ii) there is a function $\phi \colon \mathbb{R} \times [0,\infty) \to \mathbb{R}$, so that $\phi(t,\cdot)$ is continuous for fixed t, and $\|u(s)\|_E \leqslant \phi(t,s)$ for all $t \in \mathbb{R}, s \geqslant 0$ and $u \in P_t$.
Such a P is called *$\mathcal{T}$-compact* if, in addition,
(i′) P_t is closed in $C([0,+\infty); E_0)$ for every $t \in \mathbb{R}$.

Given a brochette P over $\mathcal{T}$, by $T(h)P$, $h \in \mathbb{R}$, we denote the following brochette:

$$\left(T(h)P\right)_t = T(h)P_{t-h}. \tag{3.5}$$

Definition 3.7. A brochette P over $\mathcal{T}$ is called a *pullback trajectory $\mathcal{D}$-semiattractor* (*PTSA*) for $\mathcal{H}^+$ if

(i) P is $\mathcal{T}$-compact;
(ii) $T(h)P \subset P$ for any $h \geqslant 0$ (in the sense of Definition 2.2);
(iii) P is pullback $\mathcal{D}$-attracting.

Definition 3.8. A PTSA is called a *pullback trajectory $\mathcal{D}$-attractor* (*PTA*) for $\mathcal{H}^+$ if

(ii′) $T(h)P = P$ for any $h \geqslant 0$.

Definition 3.9. A PTA is called a *minimal pullback trajectory $\mathcal{D}$-attractor* (*MPTA*) for $\mathcal{H}^+$ if it is contained (in the sense of Definition 2.2) in any other PTA. A PTSA is called a *minimal pullback trajectory $\mathcal{D}$-semiattractor* (*MPTSA*) for $\mathcal{H}^+$ if it is contained in any other PTSA.

Definition 3.10. A brochette $\mathcal{A}$ over E is called a *minimal pullback $\mathcal{D}$-attractor* (*MPA*) for the trajectory brochette $\mathcal{H}^+$ (in E_0) if

(i) $\mathcal{A}_t$ is compact in E_0 and bounded in E for each $t \in \mathbb{R}$;
(ii) for all $D \in \mathcal{D}$ and $t \in \mathbb{R}$ there is pullback attraction:

$$\sup_{u \in \mathcal{H}_\tau(D)} \inf_{v \in \mathcal{A}_t} \left\| u(t-\tau) - v \right\|_{E_0} \underset{\tau \to -\infty}{\longrightarrow} 0.$$

(iii) $\mathcal{A}$ is the minimal brochette satisfying conditions (i) and (ii) (i.e. $\mathcal{A}$ is contained in every brochette satisfying conditions (i) and (ii)).

Remark 3.11. It is obvious that MPTA, MPTSA and MPA, if they exist, are unique.

3.2. The main existence theorems

Theorem 3.12. *Assume that there exists a relatively $\mathcal{T}$-compact pullback $\mathcal{D}$-absorbing brochette P for $\mathcal{H}^+$. Then there exists an MPTA $\mathcal{U} \subset P$.*

Theorem 3.13. *If a brochette $\mathcal{P}$ is a PTSA, then there exists an MPTA $\mathcal{U}$ contained in $\mathcal{P}$.*

For a set $K \subset \mathcal{T}$, by $K(h)$, $h \geqslant 0$, we denote the set $\{v(h) \mid v \in K\}$. Similarly, for a brochette P over $\mathcal{T}$, by $P(h)$, $h \geqslant 0$, we denote the following brochette over E (the *section brochette*):

$$\big(P(h)\big)_t = \big\{v(h) \;\big|\; v \in P_t\big\}.$$

Theorem 3.14. *If a brochette $\mathcal{U}$ is an MPTA, then there is an MPA $\mathcal{A}$, and $\mathcal{A} = \mathcal{U}(0)$.*

3.3. Proofs

The proofs of the theorems require some preliminary observations.

Lemma 3.15.

(a) *For any two brochettes P_1 and P_2 over $\mathcal{T}$ satisfying the conditions* (i) *or* (ii) *of Definition* 3.7*, $P_1 \cap P_2$ also satisfies a corresponding condition.*
(b) *If P_1, P_2 are $\mathcal{T}$-compact and satisfy condition* (iii) *of Definition* 3.7*, then $P_1 \cap P_2$ also satisfies condition* (iii)*.*

Proof. Statement (a) is clear. Let us show (b). Let P_1 and P_2 be $\mathcal{T}$-compact and satisfy condition (iii). We have to show that $P_1 \cap P_2$ is a pullback $\mathcal{D}$-attracting set. If it is not so, then for some $\delta > 0$, $t \in \mathbb{R}$ and $D \in \mathcal{D}$ there is a sequence $\tau_m \to -\infty$ such that

$$\sup_{u \in \mathcal{H}_{\tau_m}(D)} \inf_{v \in (P_1 \cap P_2)_t} \left\| T(t-\tau_m)u - v \right\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in \mathcal{H}_{\tau_m}(D)$ such that

$$\inf_{v \in (P_1 \cap P_2)_t} \left\| T(t-\tau_m)u_m - v \right\|_{C([0,+\infty);E_0)} > \delta. \tag{3.6}$$

On the other hand, since P_1 and P_2 are pullback attracting, for any natural number k there exist a number m_k and elements $v_k^1 \in (P_1)_t$, $v_k^2 \in (P_2)_t$ such that

$$\big\|T(t-\tau_{m_k})u_{m_k}-v_k^1\big\|_{C([0,+\infty);E_0)}<\frac{1}{k},$$

$$\big\|T(t-\tau_{m_k})u_{m_k}-v_k^2\big\|_{C([0,+\infty);E_0)}<\frac{1}{k}.$$

Since $(P_1)_t$ is compact in $C([0,+\infty);E_0)$, without loss of generality we may assume that the sequence v_k^1 converges to an element v_0 as $k\to\infty$. Then the sequences $T(t-\tau_{m_k})u_{m_k}$ and v_k^2 also converge to v_0. Thus, $v_0\in(P_1\cap P_2)_t$ and $\|T(t-\tau_{m_k})u_{m_k}-v_0\|_{C([0,+\infty);E_0)}\underset{k\to\infty}{\longrightarrow}0$, which contradicts (3.6). □

Lemma 3.16. *Let a brochette P over $\mathcal{T}$ satisfy one of conditions* (i), (ii), (ii′) *or* (iii) *of Definitions* 3.7 *and* 3.8. *Then $T(h)P$ also satisfies a corresponding condition for all $h\geqslant 0$.*

Proof. Let P satisfy condition (ii), that is $T(s)P_{t-s}\subset P_t$ for any $s\geqslant 0$ and $t\in\mathbb{R}$. Then

$$T(s)\big(T(h)P\big)_{t-s}=T(s)T(h)P_{t-s-h}=T(h)T(s)P_{t-h-s}\subset T(h)P_{t-h}=\big(T(h)P\big)_t,$$

i.e. $T(h)P$ satisfies condition (ii). The proof of the statement of the lemma concerning condition (ii′) is similar, whereas concerning (i) it is straightforward. Let P satisfy condition (iii), that is it is pullback attracting. Since the map $T(h)$ is bounded in $C([0,+\infty);E_0)$, one has

$$\big\|T(h)u\big\|_{C([0,+\infty);E_0)}\leqslant C\|u\|_{C([0,+\infty);E_0)}$$

for some constant C and all $u\in C([0,+\infty);E_0)$. Then for any $D\in\mathcal{D}$ and $t\in\mathbb{R}$ one has

$$\begin{aligned}
&\sup_{u\in\mathcal{H}_\tau(D)}\ \inf_{v\in T(h)P_{t-h}}\big\|T(t-\tau)u-v\big\|_{C([0,+\infty);E_0)}\\
&\quad=\sup_{u\in\mathcal{H}_\tau(D)}\ \inf_{v\in P_{t-h}}\big\|T(h)\big(T(t-h-\tau)u-v\big)\big\|_{C([0,+\infty);E_0)}\\
&\quad\leqslant C\sup_{u\in\mathcal{H}_\tau(D)}\ \inf_{v\in P_{t-h}}\big\|T(t-h-\tau)u-v\big\|_{C([0,+\infty);E_0)}\underset{\tau\to-\infty}{\longrightarrow}0,
\end{aligned}$$

and, due to (3.5), $T(h)P$ is pullback attracting. □

Lemma 3.17. *An MPTSA is always an MPTA.*

Proof. Let $\mathcal{U}$ be an MPTSA. By Lemma 3.16, $T(h)\mathcal{U}$ is a PTSA for all $h\geqslant 0$, therefore $\mathcal{U}\subset T(h)\mathcal{U}$. Thus, $\mathcal{U}$ satisfies condition (ii′) from Definition 3.8, so it is a PTA, and obviously a minimal one. □

Remark 3.18. The inverse statement is also true, but is based on Theorem 3.13, which we are still going to prove; an MPTA is always an MPTSA. Really, let $\mathcal{U}$ be an MPTA and let $\mathcal{P}$ be a PTSA. By Theorem 3.13, $\mathcal{U}\subset\mathcal{P}$. Thus, $\mathcal{U}$ is contained in any PTSA, so it is an MPTSA.

Lemma 3.19. *Assume that there exists a relatively $\mathcal{T}$-compact pullback $\mathcal{D}$-absorbing brochette P for $\mathcal{H}^+$. Then there is a PTSA $\mathcal{P}\subset P$.*

Proof. For every $D\in\mathcal{D}$, $t\in\mathbb{R}$ and $\tau\leqslant\tau_0(D,t)$ one has $T(t-\tau)\mathcal{H}_\tau(D)\subset P_t$. Fix a number $t\in\mathbb{R}$, and take the closure in $C([0,+\infty);E_0)$ of the set

$$P_t^0=\bigcup_{D\in\mathcal{D}}\ \bigcup_{\tau\leqslant\tau_0(D,t)}T(t-\tau)\mathcal{H}_\tau(D),$$

and denote it by $\mathcal{P}_t$. The resulting brochette $\mathcal{P}$ is contained in P, therefore it is $\mathcal{T}$-compact. It is clear that it is pullback absorbing. It remains to show that $T(h)P^0_{t-h} \subset P^0_t$ for $h \geqslant 0$. Then the continuity of the shift operator $T(h)$ in $C([0,+\infty); E_0)$ would imply $T(h)\mathcal{P}_{t-h} \subset \mathcal{P}_t$, i.e. $T(h)\mathcal{P} \subset \mathcal{P}$. Since the function $\tau_0(D,t)$ is non-decreasing in t, we have

$$\bigcup_{D\in\mathcal{D}} \bigcup_{\tau\leqslant\tau_0(D,t-h)} T(t-\tau)\mathcal{H}_\tau(D) \subset \bigcup_{D\in\mathcal{D}} \bigcup_{\tau\leqslant\tau_0(D,t)} T(t-\tau)\mathcal{H}_\tau(D).$$

But the first union is $T(h)P^0_{t-h}$, and the second one is P^0_t. $\square$

Lemma 3.20. *(See [50, Lemma 4.2.6].) Let (X,ρ) be a metric space and $\{K_\alpha\}_{\alpha\in\Xi}$ be a system of non-empty compact sets in X. Assume that for any $\alpha_1,\alpha_2 \in \Xi$ there is $\alpha_3 \in \Xi$ such that $K_{\alpha_1} \cap K_{\alpha_2} = K_{\alpha_3}$. Then $K_0 = \bigcap_{\alpha\in\Xi} K_\alpha \neq \emptyset$, and for any $\epsilon > 0$ there is $\alpha_\epsilon \in \Xi$ such that for any $y \in K_{\alpha_\epsilon}$*

$$\inf_{x\in K_0} \rho(x,y) < \epsilon.$$

Now we can begin to prove the theorems.

Proof of Theorems 3.12 and 3.13. We need to prove Theorem 3.13, and Theorem 3.12 would then follow from Lemma 3.19.

Consider the intersection[3] $\mathcal{U}$ of all pullback trajectory $\mathcal{D}$-semiattractors for $\mathcal{H}^+$. Let us show that $\mathcal{U}$ is a PTSA. Clearly, $\mathcal{U}$ satisfies conditions (i) and (ii) of Definition 3.7. We are going to show that $\mathcal{U}$ satisfies condition (iii), i.e. it is pullback attracting.

Fix $\epsilon > 0$, $t_0 \in \mathbb{R}$ and a brochette $D \in \mathcal{D}$. In Lemma 3.20, take $X = C([0,+\infty); E_0)$, and let $\{K_\alpha\}_{\alpha\in\Xi}$ be the system of all sets $\mathcal{P}_{t_0}$ such that $\mathcal{P}$ is a PTSA for $\mathcal{H}^+$. By Lemma 3.15, an intersection of two PTSAs is a PTSA, so the intersection of any two sets from the system $\{K_\alpha\}$ belongs to this system. It is clear that

$$\mathcal{U}_{t_0} = \bigcap_{\alpha\in\Xi} K_\alpha.$$

By Lemma 3.20, there is a PTSA $\mathcal{P}_\epsilon$ such that for any $v \in (\mathcal{P}_\epsilon)_{t_0}$

$$\inf_{w\in\mathcal{U}_{t_0}} \|w - v\|_{C([0,+\infty);E_0)} < \frac{\epsilon}{2}.$$

Since $\mathcal{P}_\epsilon$ is a pullback attracting brochette, there exists τ_0 such that, for $\tau \leqslant \tau_0$,

$$\sup_{u\in\mathcal{H}_\tau(D)} \inf_{v\in(\mathcal{P}_\epsilon)_{t_0}} \left\|T(t_0-\tau)u - v\right\|_{C([0,+\infty);E_0)} < \frac{\epsilon}{2}.$$

Therefore for every $u \in \mathcal{H}_\tau(D)$ there exists $v(u) \in (\mathcal{P}_\epsilon)_{t_0}$ so that

$$\left\|T(t_0-\tau)u - v(u)\right\|_{C([0,+\infty);E_0)} < \frac{\epsilon}{2}.$$

[3] Definition 2.2 may evidently be generalized for the case of infinite number of intersecting brochettes.

We have:

$$\sup_{u\in\mathcal{H}_\tau(D)} \inf_{w\in\mathcal{U}_{t_0}} \|T(t_0-\tau)u-w\|_{C([0,+\infty);E_0)}$$
$$\leqslant \sup_{u\in\mathcal{H}_\tau(D)} \Big(\|T(t_0-\tau)u-v(u)\|_{C([0,+\infty);E_0)} + \inf_{w\in\mathcal{U}_{t_0}} \|v(u)-w\|_{C([0,+\infty);E_0)}\Big)$$
$$\leqslant \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Thus, $\mathcal{U}$ is a PTSA, being the minimal one. By Lemma 3.17, $\mathcal{U}$ is an MPTA. □

Proof of Theorem 3.14. Observe first that the invariance property $T(h)\mathcal{U}=\mathcal{U}$, $h\geqslant 0$, implies $T(h)\mathcal{U}_{t-h}=\mathcal{U}_t$, and

$$\mathcal{U}_{t-h}(h)=\mathcal{A}_t \tag{3.7}$$

for every $t\in\mathbb{R}$, where $\mathcal{A}=\mathcal{U}(0)$.

Every set $\mathcal{A}_t=\mathcal{U}_t(0)$, $t\in\mathbb{R}$, is compact in E_0 and bounded in E due to $\mathcal{T}$-compactness of $\mathcal{U}$.

Take $D\in\mathcal{D}$ and $t\in\mathbb{R}$. Since $\mathcal{U}$ is a pullback attracting brochette,

$$\sup_{u\in\mathcal{H}_\tau(D)} \inf_{v\in\mathcal{U}_t} \|T(t-\tau)u-v\|_{C([0,+\infty);E_0)} \underset{\tau\to-\infty}{\longrightarrow} 0.$$

It yields the pointwise convergence:

$$\sup_{u\in\mathcal{H}_\tau(D)} \inf_{v\in\mathcal{U}_t} \|(T(t-\tau)u-v)(h)\|_{E_0} \underset{\tau\to-\infty}{\longrightarrow} 0, \quad h\geqslant 0.$$

At $h=0$ we get

$$\sup_{u\in\mathcal{H}_\tau(D)} \inf_{v\in\mathcal{A}_t} \|u(t-\tau)-v\|_{E_0} \underset{\tau\to-\infty}{\longrightarrow} 0.$$

It remains to show that $\mathcal{A}$ is contained in every brochette A over E satisfying the property

$$\sup_{u\in\mathcal{H}_\tau(D)} \inf_{v\in A_t} \|u(t-\tau)-v\|_{E_0} \underset{\tau\to-\infty}{\longrightarrow} 0, \quad D\in\mathcal{D},\ t\in\mathbb{R}, \tag{3.8}$$

and such that A_t are compact in E_0 and bounded in E.

Define a brochette U over $\mathcal{T}$ by the formula

$$U_t=\big\{u\in\mathcal{U}_t \;\big|\; u(h)\in A_{t+h}\ \forall h\geqslant 0\big\}. \tag{3.9}$$

It suffices to show that $\mathcal{U}\subset U$. By Remark 3.18, $\mathcal{U}$ is contained in every PTSA. Hence, it is enough to show that U is a PTSA.

For any sequence $\{u_m\}\subset U_t$ converging in $C([0,+\infty);E_0)$, its limit u_0 belongs to the (closed in $C([0,+\infty);E_0)$) set $\mathcal{U}_t$. The convergence in $C([0,+\infty);E_0)$ yields the pointwise convergence: $u_m(h)\to u_0(h)$ in E_0, $h\geqslant 0$. Since A_{t+h} is compact in E_0, $u_0(h)\in A_{t+h}$, $h\geqslant 0$. Thus, each U_t is closed in $C([0,+\infty);E_0)$. Since $U\subset\mathcal{U}$, U is $\mathcal{T}$-compact. Representation (3.9) and the invariance property $T(s)\mathcal{U}=\mathcal{U}$ yield $T(s)U\subset U$, $s\geqslant 0$. It remains to show that U is a pullback attracting brochette.

If it is not so, then for some $\delta>0$, $t\in\mathbb{R}$ and $D\in\mathcal{D}$ there is a sequence $\tau_m\to-\infty$ such that

$$\sup_{u\in\mathcal{H}_{\tau_m}(D)} \inf_{v\in U_t} \|T(t-\tau_m)u-v\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in \mathcal{H}_{\tau_m}(D)$ such that

$$\inf_{v \in U_t} \left\| T(t-\tau_m)u_m - v \right\|_{C([0,+\infty);E_0)} > \delta. \tag{3.10}$$

Since $\mathcal{U}$ is pullback attracting, for any natural number k there exist a number m_k and elements $v_k \in \mathcal{U}_t$, such that

$$\left\| T(t-\tau_{m_k})u_{m_k} - v_k \right\|_{C([0,+\infty);E_0)} < \frac{1}{k}.$$

But $\mathcal{U}_t$ is compact in $C([0,+\infty);E_0)$, so without loss of generality we may assume that the sequence v_k converges to an element $v_0 \in \mathcal{U}_t$ as $k \to \infty$. Then

$$\left\| T(t-\tau_{m_k})u_{m_k} - v_0 \right\|_{C([0,+\infty);E_0)} \xrightarrow[k\to\infty]{} 0. \tag{3.11}$$

Now (3.10) and (3.11) yield $v_0 \notin U_t$, that is $v_0(s) \notin A_{t+s}$ for some $s \geqslant 0$. Using (3.8) one gets

$$\inf_{v \in A_{t+s}} \left\| u_{m_k}(t+s-\tau_{m_k}) - v \right\|_{E_0} \xrightarrow[k\to\infty]{} 0.$$

Then there is a sequence $\{v_k^*\} \subset A_{t+s}$ such that

$$\left\| T(t-\tau_{m_k})u_{m_k}(s) - v_k^* \right\|_{E_0} \xrightarrow[k\to\infty]{} 0.$$

Since A_{t+s} is compact, without loss of generality v_k^* converges to some element v^*. But (3.11) gives

$$\left\| T(t-\tau_{m_k})u_{m_k}(s) - v_0(s) \right\|_{E_0} \xrightarrow[k\to\infty]{} 0.$$

Therefore $v_0(s) = v^* \in A_{t+s}$, and we have a contradiction. □

3.4. A comparison of the concept of MPA with the pullback $\mathcal{D}$-attractors for a process

We keep assuming that we are given some spaces E, E_0 and a fixed class $\mathcal{D}$ of brochettes over E. We recall that a *process* U on E is a two-parameter family of maps

$$U(t,\tau): E \to E, \quad t,\tau \in \mathbb{R},\ t \geqslant \tau,$$

so that $U(t,t)\xi = \xi$ and $U(t,\tau)\xi = U(t,s)U(s,\tau)\xi$, for all $\xi \in E$ and $t,s,\tau \in \mathbb{R}$, $t \geqslant s \geqslant \tau$.

Definition 3.21. A brochette $\mathcal{A}$ over E is called a *pullback* $(E, E_0, \mathcal{D})$*-attractor* for U if

(i) $\mathcal{A}_t$ is compact in E_0 and bounded in E for each $t \in \mathbb{R}$;
(ii) $\mathcal{A}$ is *pullback* $(E, E_0, \mathcal{D})$*-attracting* for U, that is

$$\sup_{u \in D_\tau} \inf_{v \in \mathcal{A}_t} \left\| U(t,\tau)u - v \right\|_{E_0} \xrightarrow[\tau\to-\infty]{} 0 \tag{3.12}$$

for all $D \in \mathcal{D}$ and $t \in \mathbb{R}$;
(iii) $\mathcal{A}$ is *invariant*, i.e.

$$U(t,\tau)\mathcal{A}_\tau = \mathcal{A}_t \tag{3.13}$$

for $t,\tau \in \mathbb{R}$, $t \geqslant \tau$.

Remark 3.22. This definition is equivalent to a standard one (see e.g. [11,10]) in the case $E = E_0$. For the sake of generality, we consider the general case $E \subset E_0$, where the topology of attraction (in our case, the one of E_0) may be different from the one of the phase space E (see e.g. [4,17] for similar approaches to attractivity).

Remark 3.23. Pullback $(E, E_0, \mathcal{D})$-attractors, as defined above, can be not unique (a simple example may be found in [7]). Some minimality conditions (see e.g. [35,7]) may be added to the definition in order to provide uniqueness (we return to this issue below, in Remark 3.25).

Processes are usually generated by non-autonomous differential equations. Assume that for any $b \in E$ and $\tau \in \mathbb{R}$, Eq. (3.2) possesses a unique solution

$$u_{b,\tau} \in C\big([\tau, +\infty); E_0\big) \cap L_{\infty,\mathrm{loc}}(\tau, +\infty; E),$$

satisfying the initial condition

$$u_{b,\tau}(\tau) = b. \tag{3.14}$$

Then one can define the process U corresponding to (3.2) by the formula

$$U(t, \tau)(\xi) = u_{\xi,\tau}(t). \tag{3.15}$$

In this situation the natural family of trajectory spaces is

$$\mathcal{H}_\tau^+ = \big\{u_b \in \mathcal{T} \bigm| u_b(\cdot) = U(\cdot + \tau, \tau)b,\ b \in E\big\}, \quad \tau \in \mathbb{R}. \tag{3.16}$$

Now we examine the relation between Definitions 3.21 and 3.10.

Theorem 3.24.

(a) *If there exists a pullback $(E, E_0, \mathcal{D})$-attractor $\mathcal{A}$ for U, and $\mathcal{A} \in \mathcal{D}$, then $\mathcal{A}$ is an MPA for $\mathcal{H}^+$.*
(b) *Let the conditions of Theorem* 3.14 *hold for the trajectory brochette $\mathcal{H}^+$. If the MPTA $\mathcal{U}$ is contained in $\mathcal{H}^+$ (in the sense of Definition* 2.2*), then the MPA $\mathcal{A} = \mathcal{U}(0)$ is a pullback $(E, E_0, \mathcal{D})$-attractor for U.*

Proof. Due to the identity

$$\sup_{u \in \mathcal{H}_\tau(D)} \inf_{v \in \mathcal{A}_t} \big\|u(t-\tau) - v\big\|_{E_0} = \sup_{b \in D_\tau} \inf_{v \in \mathcal{A}_t} \big\|U(t,\tau)b - v\big\|_{E_0} \tag{3.17}$$

for all $t, \tau \in \mathbb{R}$, $t \geqslant \tau$, and $D \in \mathcal{D}$, conditions (i) (which simply coincide) and (ii), resp., of Definitions 3.10 and 3.21, are equivalent.[4] To prove (a), it remains to show that a pullback $(E, E_0, \mathcal{D})$-attractor $\mathcal{A} \in \mathcal{D}$ for U is contained in any brochette A for which axioms (i) and (ii) of Definition 3.21 hold. Fix an arbitrary number $t \in \mathbb{R}$. Since A_t is compact in E_0, for any open neighborhood W of A_t in E_0 one has $U(t,\tau)\mathcal{A}_\tau \subset W$ for all τ close to $-\infty$. If there is a point $w \in \mathcal{A}_t$ such that $w \notin A_t$, then $W_w = E_0 \backslash \{w\}$ is an open neighborhood of A_t. Therefore $w \in \mathcal{A}_t = U(t,\tau)\mathcal{A}_\tau \subset W_w$, and we arrive at a contradiction.

To check (b), we only need to show that, under the conditions of Theorem 3.14, the brochette $\mathcal{A} = \mathcal{U}(0)$ is invariant. But the inclusion $\mathcal{U} \subset \mathcal{H}^+$ and representation (3.16) yield

$$\mathcal{U}_\tau = \big\{u_b \in \mathcal{T} \bigm| u_b(\cdot) = U(\cdot + \tau, \tau)b,\ b \in \mathcal{U}_\tau(0)\big\}, \quad \tau \in \mathbb{R}. \tag{3.18}$$

[4] Of course, under assumptions (3.15) and (3.16).

Hence, for all $t \geqslant \tau$,

$$U(t,\tau)\mathcal{U}_\tau(0) = \mathcal{U}_\tau(t-\tau),$$

and by (3.7) we conclude:

$$U(t,\tau)\mathcal{A}_\tau = \mathcal{A}_t. \qquad \square$$

Remark 3.25. The above argument shows that a pullback $(E, E_0, \mathcal{D})$-attractor is in a certain sense minimal provided it belongs to the set $\mathcal{D}$. Note that the proof of this issue did not use the particular structure of the process U and is thus valid for any process. Hence, the requirement for a pullback $(E, E_0, \mathcal{D})$-attractor to belong to $\mathcal{D}$ may be a relevant alternative to minimality constraints.[5] For instance, the pullback attractors considered in [11] meet this requirement.

Remark 3.26. The referee has raised the following natural question: what is the relation of pullback trajectory attractors with pullback attractors for multivalued processes? This issue will be addressed in our forthcoming paper [42].

4. Pullback attractors for the 3D Navier–Stokes problem

4.1. Weak solutions to the 3D Navier–Stokes problem

Consider the 3D incompressible Navier–Stokes problem:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i} - \eta \Delta u + \nabla p = F, \tag{4.19}$$

$$\operatorname{div} u = 0, \tag{4.20}$$

$$u|_{\partial\Omega} = 0, \tag{4.21}$$

where u is an unknown velocity vector, p is an unknown pressure function, F is the given body force (all of them depend on a point x in a bounded domain $\Omega \subset \mathbb{R}^3$, and on a moment of time t), and $\eta > 0$ is the viscosity of a fluid.

Definition 4.1. Let $F \in L_{2,\mathrm{loc}}(0,\infty; V^*)$. A function

$$u \in L_{2,\mathrm{loc}}(0,\infty; V) \cap C_w\big([0,\infty); H\big) \cap W^1_{4/3,\mathrm{loc}}\big(0,\infty; V^*\big) \tag{4.22}$$

is an *admissible weak solution* to problem (4.19)–(4.21) if it is a weak solution, i.e.

$$\frac{d}{dt}(u,\varphi) + \eta(\nabla u, \nabla \varphi) - \sum_{i=1}^{3}\left(u_i u, \frac{\partial \varphi}{\partial x_i}\right) = \langle F, \varphi\rangle \tag{4.23}$$

for all test functions $\varphi \in V$ a.e. on $(0,\infty)$ (cf. e.g. [39]), and it satisfies the energy inequality

$$\big\|u(h)\big\|^2 \leqslant e^{-\sigma h}\left(\big\|u(0)\big\|^2 + \frac{1}{\eta}\int_0^h e^{\sigma\xi}\big\|F(\xi)\big\|^2_{V^*}\, d\xi\right) \tag{4.24}$$

[5] By the way, an artificial a posteriori procedure can insure this condition. It suffices to replace $\mathcal{D}$ with $\mathcal{D}' = \mathcal{D} \cup \{\mathcal{A}\}$, where $\mathcal{A}$ is the given $(E, E_0, \mathcal{D})$-attractor. Then $\mathcal{A}$ is an $(E, E_0, \mathcal{D}')$-attractor belonging to the set $\mathcal{D}'$.

for all $h \geqslant 0$, where

$$\sigma = \eta\lambda_1. \tag{4.25}$$

Proposition 4.2. *For every $a \in H$ and $F \in L_{2,\mathrm{loc}}(0, \infty; V^*)$, there exists an admissible weak solution to* (4.19)–(4.21) *satisfying the initial condition*

$$u|_{t=0} = a. \tag{4.26}$$

Proof. Consider a family of approximating problems: find

$$u_M \in L_2(0, M; V) \cap C\big([0, M]; H\big) \cap W_2^1\big(0, M; V^*\big), \quad u_M(0) = a,$$

so that

$$\big\langle u_M', \varphi\big\rangle + \eta(\nabla u_M, \nabla\varphi) - \sum_{i=1}^{3}\left(\frac{(u_M)_i u_M}{1 + |u_M|^2/M}, \frac{\partial\varphi}{\partial x_i}\right) = \langle F, \varphi\rangle \tag{4.27}$$

for all test functions $\varphi \in V$ a.e. on $(0, M)$, where M is a natural number. It is known [49] that such problems possess solutions.

We recall the identity (cf. [19, p. 29] or [50, Formula (6.1.21)])

$$\sum_{i=1}^{3}\left(\frac{u_i u}{1 + |u|^2/M}, \frac{\partial u}{\partial x_i}\right) = 0, \quad u \in V. \tag{4.28}$$

Substitute $2e^{\sigma t}u_M(t)$ for φ into (4.27) at a.a. $t \in (0, M)$:

$$2e^{\sigma t}\big\langle u_M'(t), u_M(t)\big\rangle = -2\eta e^{\sigma t}\big\|u_M(t)\big\|_1^2 + 2e^{\sigma t}\big\langle F(t), u_M(t)\big\rangle. \tag{4.29}$$

This implies

$$\frac{d}{dt}\big(e^{\sigma t}\big\|u_M(t)\big\|^2\big) - \sigma e^{\sigma t}\big\|u_M(t)\big\|^2 \leqslant -\eta e^{\sigma t}\big\|u_M(t)\big\|_1^2 + \frac{1}{\eta}e^{\sigma t}\big\|F(t)\big\|_{V^*}^2. \tag{4.30}$$

Integrating from 0 to $s \geqslant 0$, and taking into account (2.1) and (4.25), we get

$$e^{\sigma s}\big\|u_M(s)\big\|^2 \leqslant \|a\|^2 + \frac{1}{\eta}\int_0^s e^{\sigma\xi}\big\|F(\xi)\big\|_{V^*}^2\, d\xi. \tag{4.31}$$

Therefore, for all $h \geqslant 0$,

$$\max_{0\leqslant s\leqslant h} e^{\sigma s}\big\|u_M(s)\big\|^2 \leqslant \|a\|^2 + \frac{1}{\eta}\int_0^h e^{\sigma\xi}\big\|F(\xi)\big\|_{V^*}^2\, d\xi. \tag{4.32}$$

Due to (4.28), the solutions to (4.27) satisfy the standard bounds on $\|u_M(t)\|$ and $\int_0^t \|u_M(\xi)\|_1^2\, d\xi$ available for the weak solutions of the Navier–Stokes problem [31,39], uniformly with respect to M. Via a diagonal argument one easily concludes that there exist a subsequence u_{M_k} and a limiting function u such that $u_{M_k} \to u$ as $k \to \infty$, $M_k > T$, weakly in $L_2(0, T; V)$, weakly-$*$ in $L_\infty(0, T; H)$, and strongly in $L_2(0,T;H)$ for every $T>0$. This function u is a weak solution to (4.19)–(4.21) in class (4.22). Passing to the limit in (4.32), we get

$$\operatorname*{ess\,sup}_{0\leqslant s\leqslant h} e^{\sigma s}\|u(s)\|^2 \leqslant \|a\|^2 + \frac{1}{\eta}\int_0^h e^{\sigma\xi}\|F(\xi)\|^2_{V^*}\,d\xi. \tag{4.33}$$

This yields

$$e^{\sigma h}\|u(h)\|^2 \leqslant \|a\|^2 + \frac{1}{\eta}\int_0^h e^{\sigma\xi}\|F(\xi)\|^2_{V^*}\,d\xi, \tag{4.34}$$

e.g. by [17, Theorem 1.7, p. 33]. □

4.2. Minimal pullback attractors for 3D NS

Fix $f\in L_{2,\mathrm{loc}}(\mathbb{R};V^*)$ such that

$$\int_{-\infty}^t e^{\sigma\xi}\|f(\xi)\|^2_{V^*}\,d\xi < +\infty \tag{4.35}$$

for some (and thus for all) $t\in\mathbb{R}$. Let us construct an MPTA and an MPA for the Navier–Stokes problem (4.19)–(4.21) with $F=f$.

We take

$$E=H,$$

and

$$E_0=V_\delta^*,$$

where $\delta\in(0,1]$ is a fixed number. We define $\mathcal{D}$ as follows (cf. [11,10]). Let $\mathcal{R}$ be the set of such functions $r:\mathbb{R}\to(0,+\infty)$ that

$$\lim_{s\to-\infty} e^{\sigma s}r^2(s)=0, \tag{4.36}$$

and the function $e^{\sigma\cdot}r^2$ is increasing. The class $\mathcal{D}$ consists of the brochettes D over H for which there exist functions $r_D\in\mathcal{R}$ so that $\|w\|\leqslant r_D(t)$ for all $t\in\mathbb{R}$ and $w\in D_t$.

The trajectory spaces $\mathcal{H}^+_\tau$, $\tau\in\mathbb{R}$, are the sets of admissible weak solutions to (4.19)–(4.21) with the shifted right-hand members $F=T(\tau)f$. These trajectory spaces are contained in $\mathcal{T}$. In fact, by (4.24), every admissible weak solution u belongs to $L_{\infty,\mathrm{loc}}(0,+\infty;H)$. Since Ω is bounded, $V_\delta\subset H$ compactly, thus $H\subset V_\delta^*$ compactly. But $u'\in L_{4/3,\mathrm{loc}}(0,\infty;V^*)$, so $u\in C([0,\infty);V_\delta^*)$ by the Aubin–Simon compactness theorem [38, Corollary 4].

Theorem 4.3. *For the trajectory brochette $\mathcal{H}^+$, there exist an MPTA $\mathcal{U}$ and an MPA $\mathcal{A}=\mathcal{U}(0)$. Moreover, $\mathcal{A}\in\mathcal{D}$.*

Proof. Consider the brochette P over $\mathcal{T}$ so that the sets P_t, $t \in \mathbb{R}$, consist of functions $u \in \mathcal{T}$ satisfying the inequalities

$$\|u(h)\|^2 \leqslant 2e^{-\sigma(t+h)} R_1(t+h), \tag{4.37}$$

$$\|u'(h)\|_{V_3^*} \leqslant \eta R_2 \|u(h)\| + R_3 \|u(h)\|^2 + \|f(t+h)\|_{V_3^*} \tag{4.38}$$

for a.a. $h \geqslant 0$, where

$$R_1(s) = e^{\sigma s} + \frac{1}{\eta} \int_{-\infty}^{s} e^{\sigma\xi} \|f(\xi)\|_{V^*}^2 \, d\xi,$$

and the constants R_2 and R_3, depending only on the domain Ω, will be defined below.

By [38, Corollary 4], the sets $P_{t,M} = \{v = u|_{[0,M]}\colon\ u \in P_t\}$, $M > 0$, are relatively compact in $C([0, M]; E_0)$. This immediately implies (cf. e.g. [50, p. 183]) that P_t are relatively compact in $C([0, +\infty); E_0)$. Now it is easy to conclude that P is relatively $\mathcal{T}$-compact.

Let us check that the brochette P is pullback $\mathcal{D}$-absorbing. Fix $t \in \mathbb{R}$ and $D \in \mathcal{D}$. Set

$$\chi(s) = \max\{e^{\sigma s} r_D^2(s),\ R_1(s)\}, \quad s \in \mathbb{R}.$$

Note that the functions R_1 and χ are increasing. Thus, $\tau_0 = \chi^{-1}(R_1(t))$ is an increasing function of t (for fixed D), and $\tau_0 \leqslant t$. Let $\tau \leqslant \tau_0$. We have to show that $T(t-\tau)\mathcal{H}_\tau(D) \subset P_t$. Let $u \in \mathcal{H}_\tau(D)$, i.e. $u \in \mathcal{H}_\tau^+$ and $u(0) \in D_\tau$. Due to (4.24), for the function $v = T(t-\tau)u$ and a.a. $h \geqslant 0$, we have

$$\begin{aligned}
\|v(h)\|^2 &= \|u(t-\tau+h)\|^2 \\
&\leqslant e^{-\sigma(t-\tau+h)} \left(\|u(0)\|^2 + \frac{1}{\eta} \int_0^{t-\tau+h} e^{\sigma\xi} \|f(\xi+\tau)\|_{V^*}^2 \, d\xi \right) \\
&\leqslant e^{-\sigma(t+h-\tau)} r_D^2(\tau) + \frac{1}{\eta} \int_\tau^{t+h} e^{\sigma(\xi-t-h)} \|f(\xi)\|_{V^*}^2 \, d\xi \\
&\leqslant e^{-\sigma(t+h)} \big[\chi(\tau) + R_1(t+h)\big] \leqslant 2e^{-\sigma(t+h)} R_1(t+h),
\end{aligned} \tag{4.39}$$

since

$$\chi(\tau) \leqslant \chi(\tau_0) = R_1(t) \leqslant R_1(t+h).$$

The function u satisfies (4.23) with $F = T(\tau)f$, so v satisfies (4.23) with $F = T(t)f$. Take any function $\varphi \in V_3$. Then

$$\begin{aligned}
\big|\langle v'(h), \varphi\rangle\big| &\leqslant \eta \big|\big(v(h), \Delta\varphi\big)\big| + \sum_{i=1}^{3} \left| \left(v_i(h) v(h), \frac{\partial\varphi}{\partial x_i} \right) \right| + \big|\langle f(t+h), \varphi\rangle\big| \\
&\leqslant \big(\eta R_2 \|v(h)\| + R_3 \|v(h)\|^2 + \|f(t+h)\|_{V_3^*}\big) \|\varphi\|_{V_3},
\end{aligned} \tag{4.40}$$

with certain constants R_2 and R_3, depending only on the domain Ω. We have applied the fact of the continuous Sobolev embedding

$$V_3 \subset H_0^3(\Omega)^3 \subset W_\infty^1(\Omega)^3$$

in 3D.

Let

$$r_{\mathcal{A}}(t) = \sqrt{2e^{-\sigma t} R_1(t)}.$$

Then $r_{\mathcal{A}} \in \mathcal{R}$, and $\|w\| \leqslant r_{\mathcal{A}}(t)$ for all $t \in \mathbb{R}$ and $w \in P_t(0)$.

By Theorem 3.12 there exists an MPTA $\mathcal{U}$, and by Theorem 3.14 there is an MPA $\mathcal{A} = \mathcal{U}(0)$. Finally, since $\mathcal{A}_t = \mathcal{U}_t(0) \subset P_t(0)$, we have $\mathcal{A} \in \mathcal{D}$. □

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